Enhancing the resolving power of the Least squares Inversion with Active Constraint Balancing

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Summary

We present a new inversion approach, in which the Lagrangian multiplier is set as a variable at each parameterized blocks and automatically determined according to the parameter resolution matrix and spread function analysis. The approach, named Active Constraint Balancing (ACB), tries to balance the constraints of the optimization according to sensitivity for a given problem so that it enhances the resolution as well as the stability of the inversion process. We demonstrate the performance of the ACB by applying it to a two dimensional resistivity tomography, which results in a remarkable enhancement of the spatial resolution.

Introduction

Most of geophysical inversion schemes start from the linearization of the problem and minimization of the error in the least squares sense. Because of the non-linearity and limited data acquisition aperture, the inverse problem is usually ill-posed. To get stability and to speed up the convergence of the inverse problem, some kind of constraint or regularization is applied to the inversion process (Constable, 1987; Sasaki, 1989). Here, a great care must be taken of balancing minimization of data fitting error and regularization terms, which are the conflicting aims in the inversion, i.e., the resolution and the stability.

As far as concerning an optimization, Jupp and Vozoff (1975) developed the damped least squares inversion method with a decreasing damping factor as the process converges, and Nemeth et al. (1997) developed a dynamic smoothing approach in crosswell seismic tomography. Some studies (deGroot-Hedlin and Constable, 1990; Sasaki, 1989), on the other hand, were based on a suitable inverse model to balance the resolving power of the inversion. The blocky parameterization with varying size was used to control the resolving power of the inversion. Nevertheless, there is no quantitative or auto-matic method to optimize the resolution and the stability.

Parameter resolution matrix and Spread function

It can be shown that the iterative solution of the non-linear geophysical inverse problem reduces to the matrix equation (Jupp and Vozoff, 1975)

$$\mathbf{e} = \mathbf{J}\Delta\mathbf{p} \,. \qquad (1)$$

Here $\mathbf{e}$ is the error or discrepancy vector between the observed and calculated data for the initial model vector, $\mathbf{p_0}$, $\mathbf{J}$ is the partial derivative or Jacobian matrix, and $\Delta\mathbf{p}$ is the model perturbation vector. To find optimum model perturbations $\Delta\mathbf{p}$, we minimize the following objective function

$$S = \mathbf{e}^T\mathbf{e} + \lambda\{(\partial^n \Delta\mathbf{p})^T(\partial^n \Delta\mathbf{p})\} \,, \qquad (2)$$

where λ is the Lagrangian multiplier. The first term in the right hand side of the equation (2) means the minimization of the error in a least squares sense, and the second term incorporates the constraint or regularization about the solution to be obtained. If *n* equals 0, above equation corresponds to the Marquardt-Levenberg method, while it is the smoothness-constrained inversion when *n* equals 1 or 2. Minimizing the above equation with respect to the model perturbation vector yields the following normal equation

$$[\mathbf{J}^T\mathbf{J} + \lambda(\partial^n)^T\partial^n]\Delta\mathbf{p} = \mathbf{J}^T\mathbf{e} \,. \qquad (3)$$

And model perturbation vector can be obtained by the inversion of the matrix in the left bracket of equation (3)

$$\Delta\mathbf{p} = \mathbf{J}^+\mathbf{e} \,, \qquad (4)$$

where $\mathbf{J}^+$ is the pseudo inverse matrix

$$\mathbf{J}^+ = [\mathbf{J}^T\mathbf{J} + \lambda(\partial^n)^T\partial^n]^{-1}\mathbf{J}^T \,. \qquad (5)$$

Let's introduce the constraint operator matrix or parameter resolution matrix **R** (Jackson, 1972; Menke, 1984)

$$\mathbf{R} = \mathbf{J}^+\mathbf{J} \,, \qquad (6)$$

This can be regarded as a weighted average filter acting on the true model perturbation vector. This fact can be visualized as

$$\Delta\mathbf{p} = \mathbf{R}\mathbf{J}^{-1}\mathbf{e} = \mathbf{R}\Delta\mathbf{p}_g \,, \qquad (7)$$

where $\Delta\mathbf{p}_g$ is the true model perturbation vector which will be obtained if the inverse of the Jacobian $\mathbf{J}^{-1}$ exists. Since the parameter resolution matrix **R** is the product of Jacobian and the pseudo inverse matrix, we can identify whether certain parameter can be resolved or not. If one parameter can be almost perfectly resolved, corresponding row vector of the parameter resolution matrix should have the value of unity for that parameter and zero for others. On the other hand, if a parameter cannot be resolved at all, the row vector consists of random numbers without dominating one.

As a simple extension of the parameter resolution matrix, another criteria of resolving power can be derived. Although parameter resolution matrix shows the sensitivity of the parameters, it cannot give quantitative measure of the goodness of resolution for intermediate range between perfect and very poor. To quantify the resolving power, we use the Backus-Gilbert spread function (Menke, 1984), which evaluates the spatial distribution of the row vectors of the parameter resolution matrix. A large value of spread function of a certain parameter means that the resolving power of that parameter is degenerated, or vice versa. This spread function is to be calculated for i th parameter as

$$S_i = \sum_{j=1}^{N} w_j \{(1 - C_{ij}) R_{ij}\}^2 , \qquad (8)$$

where C_{ij} is the matrix incorporated to account for the constraint or regularization (i.e., the effect of the damping or smoothness-constraint operator) in the inversion, and w_{ij} is the weighting factor for which we use the spatial distance between two parameters i and j.

Active Constraint Balancing (ACB)

In the inverse modeling, we try to choose a good estimate of the Lagrangian multiplier, but practically it is not simple to get the best one. A large Lagrangian multiplier, in principle, would make more constraint or regularization to the solution and give poor resolution about the parameters. On the other hand, too small Lagrangian multiplier may cause the solution unstable. An intermediate value must be chosen, compromising the resolution and the stability. However, this approach neglects the fact that each parameter is not to be equally resolvable. For unresolvable parameter given Lagrangian multiplier can be too small, so that artifact can grow. For the highly resolvable one, on the other hand, resolution is to be degraded and details of the earth cannot be recoverable.

Hence, one may come to use varying Lagrangian multipliers to get both the resolution and the stability. We use the ACB to determine the spatially varying Lagrangian multipliers. In the ACB, we first calculate the spread functions of the parameter resolution matrix with a fixed small (for example, 0.005) Lagrangian multiplier in equation (5). Next, we convert it to the variable Lagrangian multiplier set through pre-specified condition. If the spread function of a parameter is large, which is the case of poor resolving power, the ACB assigns a large value of Lagrangian multiplier to that parameter, or vice versa. The pre-specified condition is a decision-making process whether resolving power for the parameter is high or not. According to the spread functions, Lagrangian multipliers are to be set linearly in logarithmic space in the interval of pre-selected lower and upper limit values.

This approach may have the similar meaning to the blocky parameterization with varying size in non-linear inversion. Although these approaches can account for the differing resolutions of parameters in advance, they cannot reflect the varying conditions encountered during the iterative inversion process. The ACB, on the other hand, can actively reflect the characteristics of the model and data in the process of inversion.

Numerical example

We applied the ACB to the cross-hole DC resistivity inversion (i.e. resistivity tomography) for the synthetic data set. Our resistivity tomography algorithm consists of a 2.5 dimensional finite difference modeling and the smoothness constrained least squares inversion. The Jacobian is calculated using the reciprocity principle. The synthetic data set is an output from three dimensional resistivity modeling using integral equation technique with the same approach as used in Beasley and Ward (1986). The forward model is a simple earth model with two conductive objects centered on the section as shown in Figure 1. The resistivities of the objects are 10 ohm-m, while that of half space is 100 ohm-m. The lengths of objects along the strike direction are 60m and the objects were divided into 5 × 1 × 15 cells in the modeling. The section to be imaged was uniformly divided with the size of 2m × 2m.

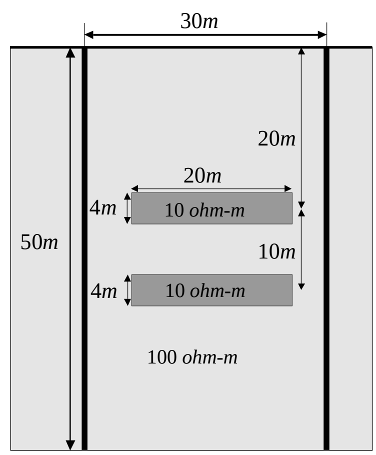

Fig. 1 Synthetic model used in the resistivity tomography. Two conductive objects are present at the center of the section.

In this example, we generated three kinds of data sets; cross-hole, inline, and hole-to-surface data. The data acquisition geometry in cross-hole configuration is a modified pole-dipole array as shown in Figure 2. In the modified pole-dipole array, the positive current electrode is located in the transmitting borehole while the fixed negative current electrode being on the surface near the receiving hole instead of distant earthing. This electrode array was used to overcome the negative sensitivity problem in the conventional electrode arrays. In the cross-hole survey using the conventional electrode arrays, the high apparent resistivity anomalies would be shown in the presence of the conductive targets being two dimensional or elongated along the strike direction. Furthermore, this electrode array produces the relatively higher voltage than the conventional pole-dipole array. The conventional pole-dipole array, of which the negative current electrode is earthed distantly, was used for the inline and the hole-to-surface measurement.

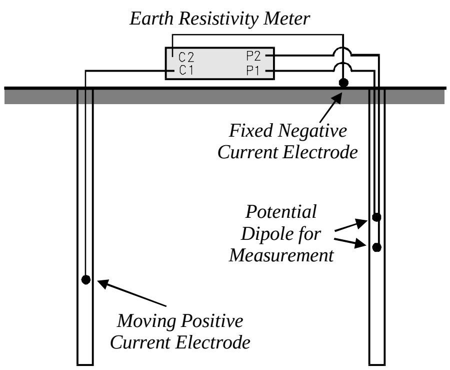

Fig. 2 A schematic presentation of the modified pole-dipole array for resistivity tomography measurement.

First, we calculated the parameter resolution matrix and spread function with three different data set combinations. The data set combinations were (a) cross-hole, inline and hole-to-surface data, (b) cross-hole and inline data, and (c) cross-hole and hole-to-surface data. Figure 3 shows the spatial distributions of the spread functions for three kinds of data set combinations. These spatial distributions of spread functions were calculated based on the same inverse model of 100 ohm-m half space.

We can see relatively high values of spread function in the bottoms of the sections where the resolving power of geotomography is worse. Furthermore, the spatial distribution of spread function varies with the data set combinations. The data set combination (a), where all the possible data sets were used, shows relatively low values of spread function so that higher resolving power than other configurations is expected. Unlike the seismic or radar tomography that uses wave propagation phenomenon, the central part of the section has lower resolving power while higher resolving power is to be shown at the vicinity of electrode locations. Therefore, near surface area shows poor resolving power without hole-to-surface data set as shown in (b). Moreover, inline data set is crucial to enhance the resolving power near the borehole

The ACB generates the optimized spatially varying Lagrangian multipliers through spread function analysis. We calculated the distributions of Lagrangian multipliers based on the spread function distribution shown in Figure 3. In this example, the pre-specified upper and lower limits of Lagrangian multipliers are 1 and 0.005. The resultant spatial distributions of Lagrangian multipliers are shown in Figure 4 for the three different data set combinations. We can easily identify the fact that these spatial distributions of Lagrangian multipliers are simple conversion of those of spread functions.

To visualize the enhanced resolution through ACB, we performed resistivity tomography with ACB and fixed Lagrangian multiplier, a conventional approach. The used value of fixed Lagrangian multiplier was 0.05. In this example, data set configuration (a) was used. Comparing the reconstructed images in Figure 5, the objects can be seen much clearer and the artifact in the bottom of the section is relaxed in the case of the ACB approach.

Conclusion

We have developed a new inversion algorithm to enhance the resolving power of least squares inversion through introducing the varying Lagrangian multipliers. We analyzed the goodness of resolution in the inversion through the parameter resolution matrix and the spread function, and could determine an optimized spatial distribution of Lagrangian multipliers to enhance the resolving power of the inversion. In principle, this ACB approach is applicable to a variety of other least squares inverse problems.

Acknowledgement

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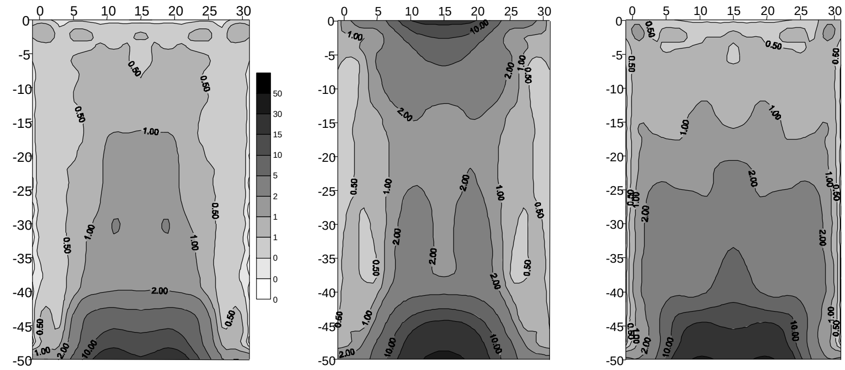
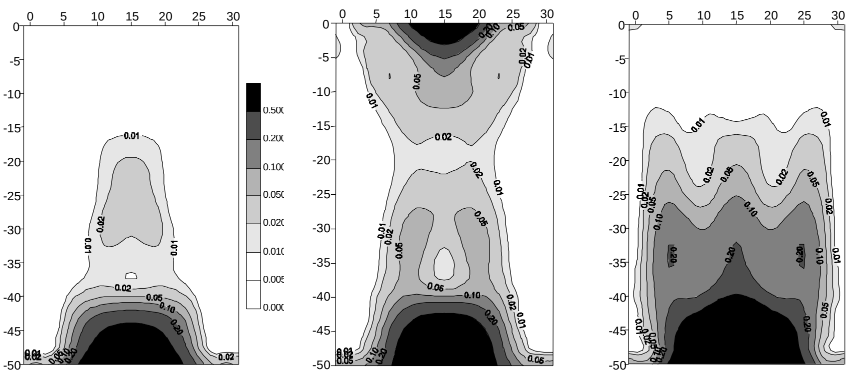
Inversion with Active Constraint Balancing



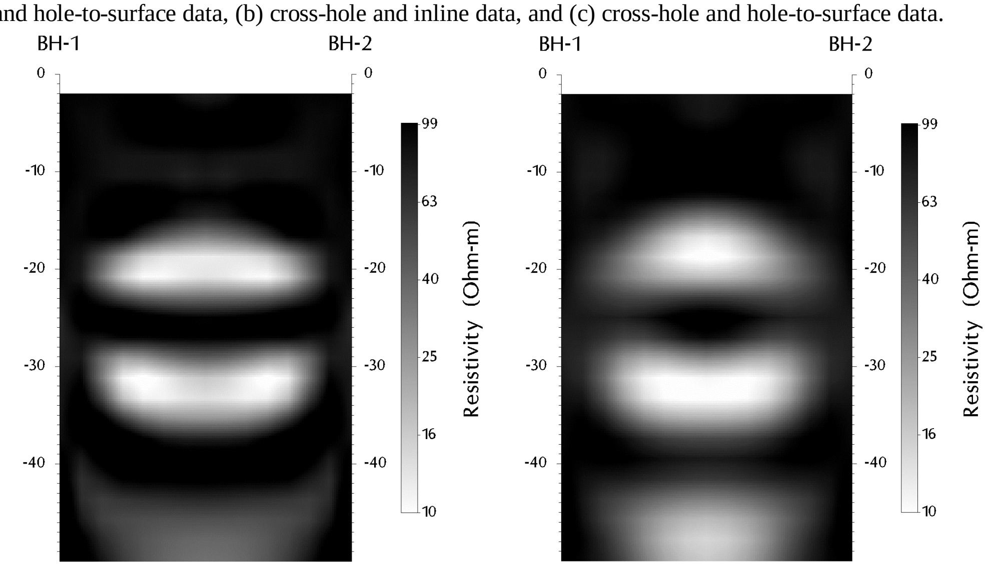
(a) (b) (c)

Fig. 3 Spatial distribution of spread function, based on the model of 100 ohm-m half space for different data sets; (a) cross-hole, inline and hole-to-surface data, (b) cross-hole and inline data, and (c) cross-hole and hole-to-surface data.



(a) (b) (c)

Fig. 4 Spatial distribution of Lagrangian multipliers derived from the spread function distribution shown in Figure 2; (a) cross-hole, inline and hole-to-surface data, (b) cross-hole and inline data, and (c) cross-hole and hole-to-surface data.



(a) (b)

Fig. 5 Tomographic images (a) by the full optimization of the Lagrangian multiplier distribution through the Active Constraint Balancing, and (b) by a fixed Lagrangian multiplier of 0.05, a conventional approach. Data set configuration (a) in Figures 3 and 4 was used in the inversion.